we obtain

$$\begin{vmatrix} A_{11}+\gamma_1 & A_{12}\cdots & A_{1m} & \alpha_1 \\ A_{21} & A_{22}+\gamma_2\cdots & A_{2m} & \alpha_2 \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2}\cdots & A_{mm}+\gamma_m & \alpha_m \\ \alpha_1 & \alpha_2\cdots & \alpha_m & \frac{2}{K}\left(\frac{1}{B}-\log a\right)+4\log\frac{\pi}{2} \end{vmatrix} = 0, \quad (23)$$

since

$$\int_0^\infty e^{-2t}\frac{\tanh t}{t}dt = \log\frac{\pi}{2}\cdot$$

The quantities α_n, A_{np} can be easily computed from Eqs. (19), (20) and (16) with the help of the result[2]

$$\int_0^\infty (2t)^n e^{-2t}\tanh t dt = n!\left\{\left(1-\frac{1}{2^n}\right)\zeta(n+1)-\tfrac{1}{2}\right\},$$

where $\zeta(n)$ is the Riemann Zeta-function, tabulated for integral n in J. Edwards, "The integral calculus," vol. 2, Macmillan, London, 1922, p. 144.[3]

The solution for $K=1$ differs from the well-known exact solution for this case by less than 0.2 per cent, when only the first three of b_n are retained, provided that $a\leq\frac{1}{2}$. For larger values of K and a it may be necessary to retain more terms to achieve the desired accuracy but for practical values the amount of computation required is not excessive.

[2] When $n=0$, the result reduces to $\int_0^\infty e^{-2t}\tanh t dt = \log 2-\frac{1}{2}$.

[3] A four-figure table is given in E. Jahnke and F. Emde, *Tables of functions*, Dover Publications, New York, 1943, p. 273.

LARGE DEFLECTION OF CANTILEVER BEAMS*

By K. E. BISSHOPP and D. C. DRUCKER (*Armour Research Foundation*)

The solution for large deflection of a cantilever beam[1] cannot be obtained from elementary beam theory since the basic assumptions are no longer valid. Specifically the elementary theory neglects the square of the first derivative in the curvature formula and provides no correction for the shortening of the moment arm as the loaded end of the beam deflects. For large finite loads, it gives deflections greater than the length of the beam! The square of the first derivative and correction factors for the shortening of the moment arm become the major contribution to the solution o

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[1] This problem was considered by H. J. Barten, "On the Deflection of a Cantilever Beam," Quarterly of Applied Math., **2,** 168–171 (1944). Previously an approximate solution had been obtained by Gross und Lehr in *Die Federn*, Berlin VDI Verlag, 1938.

large deflection problems. The following theory which utilizes these corrections is in agreement with experimental observations.

The derivation is based on the fundamental Bernoulli-Euler theorem which states that the curvature is proportional to the bending moment. It is assumed also that bending does not alter the length of the beam.

Considering a long, thin cantilever leaf spring, let L denote the length of beam, Δ the horizontal component of the displacement of the loaded end of the beam, δ the corresponding vertical displacement, P the concentrated vertical load at the free end, B the flexural rigidity, that is $B=EI$, when cross-sectional dimensions are of the

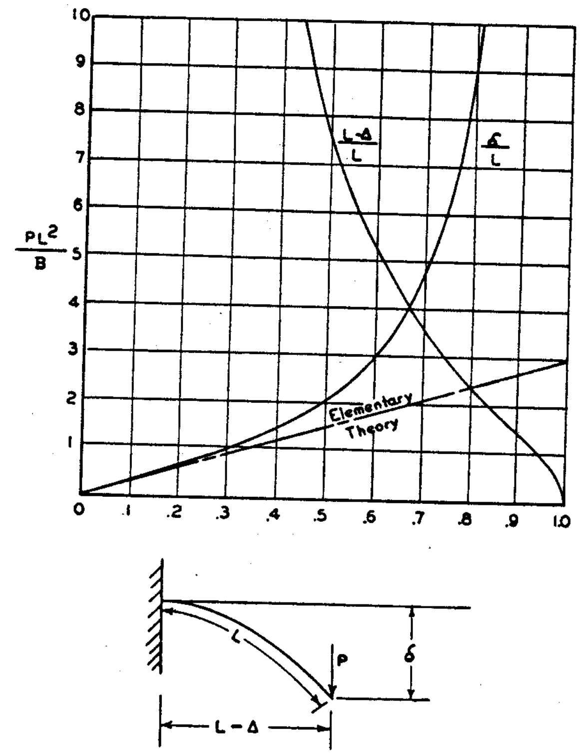

FIG. 1.

same order of magnitude, and $B=EI/(1-\nu^2)$ for "wide" beams, where ν is the Poisson ratio. The exact expression for the curvature of the elastic line may be stated conveniently in terms of arc length and slope angle denoted by s and ϕ, respectively, so that if x is the horizontal coordinate measured from the fixed end of the beam, the product of B and the curvature of the beam equals the bending moment M:

$$B\frac{d\phi}{ds} = P(L - x - \Delta) = M \tag{1}$$

or

$$\frac{d^2\phi}{ds^2} = -\frac{P}{B}\frac{dx}{ds} = -\frac{P}{B}\cos\phi, \tag{2}$$

whence

$$\frac{1}{2}\left(\frac{d\phi}{ds}\right)^2 = -\frac{P}{B}\sin\phi + C. \tag{3}$$

The constant C can be evaluated directly by observing that the curvature at the loaded end is zero. Then if ϕ_0 is the corresponding angle of slope

$$\frac{d\phi}{ds} = \sqrt{\frac{2P}{B}}(\sin\phi_0 - \sin\phi)^{1/2}. \tag{4}$$

The value of ϕ_0 cannot be found directly from this equation but it is implied by the requirement that the beam be inextensible, so that

$$\sqrt{\frac{2P}{B}}\int_0^L ds = \int_0^{\phi_0}(\sin\phi_0 - \sin\phi)^{-1/2}d\phi = \sqrt{2}\left(\frac{PL^2}{B}\right)^{1/2}. \tag{5}$$

In order to evaluate this elliptic integral, denote PL^2/B by α^2 and let

$$1 + \sin\phi = 2k^2\sin^2\theta = (1 + \sin\phi_0)\sin^2\theta. \tag{6}$$

Then

$$\alpha = \int_{\theta_1}^{\pi/2}(1 - k^2\sin^2\theta)^{-1/2}d\theta, \qquad \sin\theta_1 = \sqrt{2}/2k. \tag{7}$$

The next step is to represent the deflection δ in terms of α and an elliptic integral. Since

$$\frac{dy}{d\phi}\cdot\frac{d\phi}{ds} = \frac{dy}{ds} = \sin\phi,$$

and since we have $d\phi/ds$ from Eq. (4),

$$\frac{dy}{d\phi}\sqrt{\frac{2P}{B}}(\sin\phi_0 - \sin\phi)^{1/2} = \sin\phi.$$

Thus

$$\delta = \int_0^y dy = \sqrt{\frac{B}{2P}}\int_0^\phi \frac{\sin\phi d\phi}{(\sin\phi_0 - \sin\phi)^{1/2}}.$$

With the aid of Eq. (6) we obtain

$$\frac{\delta}{L} = \frac{\sqrt{2}}{2\alpha}\int_0^{\phi_0}\frac{\sin\phi d\phi}{(\sin\phi_0 - \sin\phi)^{1/2}} = \frac{1}{\alpha}\int_{\theta_1}^{\pi/2}\frac{(2k^2\sin^2\theta - 1)d\theta}{(1 - k^2\sin^2\theta)^{1/2}}.$$

This equation can be split up into complete and incomplete elliptic integrals of the first and second kinds. In the notation of Jahnke and Emde,

$$\frac{\delta}{L} = \frac{1}{\alpha}[F(k) - F(k,\theta_1) - 2E(k) + 2E(k,\theta_1)], \tag{8}$$

$$\alpha = F(k) - F(k,\theta_1),$$

so that

$$\frac{\delta}{L} = 1 - \frac{2}{\alpha}[E(k) - E(k,\theta_1)]. \tag{9}$$

The horizontal displacement of the loaded end is calculated from Eqs. (1) and (4) with $x=0$ when $\phi=0$. Thus

$$P(L-\Delta) = B\left(\frac{d\phi}{ds}\right)_{\phi=0} = B\sqrt{\frac{2P}{B}}\,(\sin\phi_0)^{1/2}$$

or

$$\frac{L-\Delta}{L} = \frac{\sqrt{2}}{\alpha}\,(\sin\phi_0)^{1/2}. \tag{10}$$

From Eq. (6) we have $\sin\phi_0 = 2k^2-1$.

Numerical results can be obtained by: (1) selecting values of k corresponding to tabulated values of the modular angle in the elliptic function tables and (2) determining θ_1 and α from Eq. (7). After this has been done, δ/L and $(L-\Delta)/L$ can be calculated from Eqs. (9) and (10) and plotted against $\alpha^2 = PL^2/B$. The results of these calculations are shown in Fig. 1.

CORRECTIONS TO MY PAPER

ON THE DEFLECTION OF A CANTILEVER BEAM*

QUARTERLY OF APPLIED MATHEMATICS, 2, 168–171 (1944)

BY H. J. BARTEN

This paper is correct up to the equation

$$\theta_L = \int_0^L as\cos\theta\,ds.$$

The next step

$$\frac{d\theta_L}{dL} = aL\cos\theta_L$$

is incorrect since θ is not only a function of L, but is also a function of s. This error makes Eqs. (9), (11), and (12) incorrect.

Using the relation

$$\frac{d\theta}{ds} = a(x_L - x)$$

and the various steps used in the original paper, we find that

$$a^{1/2}L = F\left(k, \frac{\pi}{2}\right) - F(k, \delta).$$

By using δ as an independent variable we can calculate corresponding values of k and

* Received June 25, 1945.