

Large and small deflections of a cantilever beam

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Abstract

The classical problem of the deflection of a cantilever beam of linear elastic material, under the action of an external vertical concentrated load at the free end, is analysed. We present the differential equation governing the behaviour of this physical system and show that this equation, although straightforward in appearance, is in fact rather difficult to solve due to the presence of a non-linear term. In this sense, this system is similar to another well known physical system: the simple pendulum. An approximation of the behaviour of a cantilever beam for small deflections was obtained from the equation for large deflections, and we present various numerical results for both cases. Finally, we compare the theoretical results with the experimental results obtained in the laboratory.

1. Introduction

In this paper we shall analyse an example of a simple physical system, the deflections of a cantilever beam. We shall see that it is not complicated to formulate the equations governing its behaviour or to study it in a physics laboratory at university level. However, a differential equation with a non-linear term is also obtained. Moreover—as occurs with the simple pendulum for small oscillations [1]—when small deflections of the cantilever beam are considered, it is possible to find a simple analytical solution to the problem. In this sense, the study of large and small deflections of a cantilever beam presents a certain analogy with the study of large and small oscillations of a simple pendulum.

The mathematical treatment of the equilibrium of cantilever beams does not involve a great degree of difficulty [2–4]. Nevertheless, unless small deflections are considered, an analytical solution does not exist, since for large deflections a differential equation with a non-linear term must be solved. The problem is said to involve geometrical non-linearity [5, 6]. An excellent

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treatment of the problem of the deflection of a beam, built-in at one end and loaded at the other with a vertical concentrated force, can be found in *The Feynman Lectures on Physics* [2], as well as in other university textbooks on physics, mechanics and the elementary strength of materials. However, in these books the discussion is limited to the consideration of small deflections and they present a formula for the vertical deflection of the end free of the cantilever beam that shows a relation of proportionality between this deflection and the external force applied [2, 4]. The analysis of large deflections of these types of cantilever beams of elastic material can be found in Landau's book on elasticity [5], and the solution in terms of elliptic integrals was obtained by Bisshopp and Drucker [7]. Nevertheless, the developments presented in these last references are difficult for first year university students.

In this paper we analyse the problem of the deflection of a cantilever beam, in the case of both large and small deflections. Firstly, we obtain the differential equation for the deflection curve (elastic curve) in the general case of large deflections, as well as the equations that determine the Cartesian coordinates of each point on the elastic curve. These equations will be solved numerically in an easy way with the aid of the program Mathematica, without using elliptic functions. We obtain the approximation for small deflections from the equations presented for large deflections. Various general numerical results are presented and, only by way of example, these are compared with the experimental results obtained in the laboratory.

2. Theoretical analysis

We consider a long, thin, cantilever beam of uniform rectangular cross section made of a linear elastic material that is homogeneous and isotropic, in which the law of behaviour of the material is represented by the linear relation

$$\sigma = \varepsilon E \quad (1)$$

which is known as Hooke's law [2,8], and where σ is the normal stress, ε is the strain and E is the modulus of elasticity or Young's modulus [2]. The deflection of a cantilever beam is essentially a three-dimensional problem. An elastic stretching in one direction is accompanied by a compression in the perpendicular directions: the ratio is known as Poisson's ratio. However, we can ignore this effect when the length of the beam is larger than the thickness of the perpendicular cross section and this is shorter than the curvature radius of the beam [2]. In this study, we assume that the beam is non-extensible and the strains remain small, and that the Bernoulli–Euler hypothesis is valid, i.e. plane cross sections which are perpendicular to the neutral axis before deformation remain plane and perpendicular to the neutral axis after deformation. Next, we also assume that the plane sections do not change their shape or area. Following, for instance, the analysis proposed by Feynman regarding the study of the deflection of a cantilever beam, it is possible to write the Bernoulli–Euler bending moment–curvature relationship for a uniform-section rectangular beam of linear elastic material as follows [2]:

$$EI \frac{d\varphi}{ds} = M \quad (2)$$

where M and $\kappa = d\varphi/ds$ are the bending moment and the curvature at any point of the beam, respectively, and I is the moment of inertia of the beam cross section about the neutral axis [2–4]. We will consider the deflections of a cantilever beam subjected to one vertical concentrated load at the free end by supposing that the deflection due to its self-weight is zero. This implies considering a massless beam [2].

2.1. Large deflections of a cantilever beam

Figure 1 shows a cantilever beam of length L with a concentrated load F applied at the free end of the beam. In this figure δ_x and δ_y are the horizontal and vertical displacements at the free end, respectively, and φ_0 takes into account the maximum slope of the beam. We take the

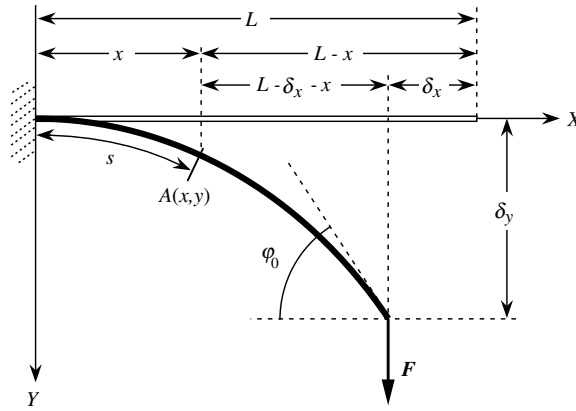


Figure 1. Cantilever beam loaded with an external vertical concentrated load at the free end and definitions of the parameters for large deflections.

origin of the Cartesian coordinate system at the fixed end of the beam and let (x, y) be the coordinates of point A, and s the arc length of the beam between the fixed end and point A. If we differentiate equation (2) once with respect to s , we obtain

$$EI \frac{d^2\varphi}{ds^2} = \frac{dM}{ds} \tag{3}$$

where the bending moment M at a point A with Cartesian coordinates (x, y) is given by the equation (figure 1)

$$M(s) = F(L - \delta_x - x). \tag{4}$$

By differentiating equation (4) once with respect to s , taking into account the relation $\cos \varphi = dx/ds$ and substituting in equation (3), we obtain the non-linear differential equation that governs the deflections of a cantilever beam made of a linear material under the action of a vertical concentrated load at the free end:

$$EI \frac{d^2\varphi}{ds^2} + F \cos \varphi = 0. \tag{5}$$

Equation (5), although straightforward in appearance, is in fact rather difficult to solve because of the non-linearity inherent in the term $\cos \varphi$. As indicated in the introduction, this situation is similar to that which can be found in the study of the simple pendulum [1]. In order to obtain the solution of equation (5), this equation is multiplied by $d\varphi/ds$, so that it becomes

$$EI \frac{d\varphi}{ds} \frac{d^2\varphi}{ds^2} + F \cos \varphi \frac{d\varphi}{ds} = 0 \tag{6}$$

which can be written as

$$\frac{d}{ds} \left[\frac{1}{2} EI \left(\frac{d\varphi}{ds} \right)^2 + F \sin \varphi \right] = 0. \tag{7}$$

Equation (7) is immediately integrable, taking into account that at the free end $\varphi(L) = \varphi_0$, where φ_0 is the unknown slope at the free end of the beam (see figure 1), and from equations (2) and (4), it follows that $(d\varphi/ds)_{s=L} = 0$. From equation (7) we can obtain

$$\left(\frac{d\varphi}{ds} \right)^2 = \frac{2}{EI} (F \sin \varphi_0 - F \sin \varphi) \tag{8}$$

and by integrating it we can obtain the following equation for the arc length s as a function of the slope φ :

$$s = \sqrt{\frac{EI}{2F}} \int_0^\varphi \frac{d\varphi}{\sqrt{\sin \varphi_0 - \sin \varphi}}. \quad (9)$$

The total length L corresponds to the unknown angle φ_0 at the free end of the beam:

$$L = \sqrt{\frac{EI}{2F}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\sin \varphi_0 - \sin \varphi}}. \quad (10)$$

Equation (10) allows us to obtain the angle φ_0 at the free end of the beam as a function of the length, L , the modulus of elasticity, E , the moment of inertia of the cross section of the beam, I , and the external load, F . After obtaining the arc length s as a function of φ (equation (9)), and taking into account that $\cos \varphi = dx/ds$ and $\sin \varphi = dy/ds$, the x and y coordinates of the horizontal and vertical deflection at any point along the neutral axis of the cantilever beam are found as follows:

$$x = \sqrt{\frac{2EI}{F}} \left(\sqrt{\sin \varphi_0} - \sqrt{\sin \varphi_0 - \sin \varphi} \right) \quad (11)$$

$$y = \sqrt{\frac{EI}{2F}} \int_0^\varphi \frac{\sin \varphi d\varphi}{\sqrt{\sin \varphi_0 - \sin \varphi}}. \quad (12)$$

From figure 1, it is easy to see that the horizontal and vertical displacements at the free end can be obtained from equations (11) and (12) taking $\varphi = \varphi_0$:

$$\delta_x = L - x(\varphi_0) \quad (13)$$

$$\delta_y = y(\varphi_0). \quad (14)$$

We introduce the non-dimensional load parameter α and the non-dimensional coordinates ξ and η defined as follows:

$$\alpha = \frac{FL^2}{2EI} \quad (15)$$

$$\xi = x/L \quad \eta = y/L \quad (16)$$

as well as the non-dimensional tip deflection ratios:

$$\beta_x = \delta_x/L = 1 - \xi(\varphi_0) \quad \beta_y = \delta_y/L = \eta(\varphi_0). \quad (17)$$

Using the parameter α , it is possible to obtain a more general view of the results, because cantilever beams with different combinations of E , I , F and L may give the same value of α and, consequently, they would have the same behaviour. Taking into account the definitions of α , ξ and η , equations (10)–(12) can be written as

$$\int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\sin \varphi_0 - \sin \varphi}} - 2\sqrt{\alpha} = 0 \quad (18)$$

$$\xi = \frac{1}{\sqrt{\alpha}} \left(\sqrt{\sin \varphi_0} - \sqrt{\sin \varphi_0 - \sin \varphi} \right) \quad (19)$$

$$\eta = \frac{1}{2\sqrt{\alpha}} \int_0^\varphi \frac{\sin \varphi d\varphi}{\sqrt{\sin \varphi_0 - \sin \varphi}}. \quad (20)$$

Equation (18) allows us to obtain φ_0 as a function of the non-dimensional load parameter α . However, equations (19) and (20) are elliptic integrals that may be evaluated numerically [7].

2.2. Small deflections of a cantilever beam

We obtain the approximate solution for small deflections of the cantilever beam from the general equations in the previous section instead of formulating the problem again from the beginning, as is done, for instance, in Feynman's book [2]. In the case of small deflections, the angle φ is small for all the points on the cantilever beam. If φ and φ_0 are sufficiently small, the approximations $\sin \varphi \approx \varphi$ and $\sin \varphi_0 \approx \varphi_0$ may be used in equations (18)–(20). From equation (18), we can write

$$\int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\varphi_0 - \varphi}} - 2\sqrt{\alpha} = 0 \quad (21)$$

and integrating it, we can easily see that, for small deflections, the value of the angle at the free end of the beam, φ_0 , is equal to the load parameter, α , that we introduced in the previous section, and taking into account the definition of α (equation (15)):

$$\varphi_0 = \frac{FL^2}{2EI}. \quad (22)$$

From equation (22), we can see that, if α is sufficiently small, it is possible to consider small deflections for cantilever beams loaded with a vertical concentrated force at the free end. We can conclude that, provided the approximation $\sin \alpha \approx \alpha$ can be made, it is possible to consider small deflections. For small deflections, equation (19) takes the form

$$\xi = 1 - \sqrt{1 - \frac{\varphi}{\alpha}} \quad (23)$$

and we can see that, for $\varphi = \varphi_0 = \alpha$, we obtain $\xi(\varphi_0) = 1$, that is, $x = L$ and then $\delta_x = 0$. This implies that, under the hypothesis of small deflections, there is no horizontal displacement at the free end (figure 2). From equation (4) it can be deduced that, for small deflections, the bending moment is

$$M(s) = F(L - x) \quad (24)$$

which is the equation Feynman uses as the starting point in his study [2]. Finally, for small deflections, from equation (20) it is easy to obtain the relation

$$\eta = \frac{2}{3} \left[\alpha - (\varphi + 2\alpha) \sqrt{1 - \frac{\varphi}{\alpha}} \right]. \quad (25)$$

The vertical displacement at the free, β_y , can be obtained considering $\varphi = \varphi_0 = \alpha$ in equation (25):

$$\beta_y = \frac{2}{3}\alpha. \quad (26)$$

Using equations (23) and (25), it is possible to write the following non-dimensional equation for the elastic curve of the cantilever beam for small deflections:

$$\eta = 2\alpha \left(\frac{\xi^2}{2} - \frac{\xi^3}{6} \right). \quad (27)$$

It is easy to check that, if we substitute the values of ξ , η and α from equations (15) and (16) in (27), we obtain the equation for the elastic curve for small deflections which appears as equation (38.41) in [2].

3. Numerical results

As mentioned above, in order to study large deflections of a cantilever beam subjected to a vertical concentrated load at the free end, it is necessary to know the angle φ_0 . To do this, it is necessary to solve equation (18) in order to obtain φ_0 as a function of α . Instead of writing equations (18) and (20) in terms of elliptic functions, by means of complex changes of variable

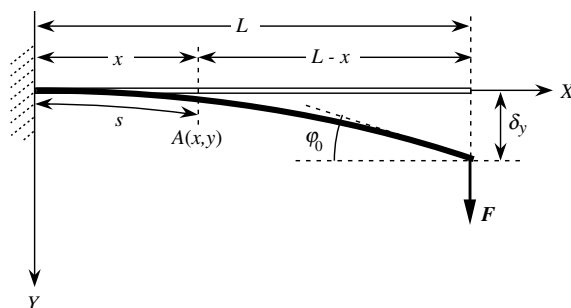


Figure 2. Cantilever beam loaded with an external vertical concentrated load at the free end for small deflections.

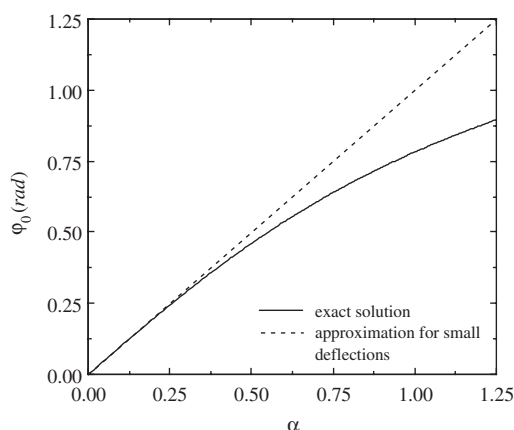


Figure 3. Values obtained from equation (18) for φ_0 as a function of the non-dimensional load parameter α .

which give little insight as to the nature of the solutions, it is easier for students to solve them numerically using one of the commercial software packages available. Nowadays all university students of science and engineering are familiar with programs such as Mathematica or Matlab, which also have student versions. Taking this into account, we used the Mathematica program, with the aid of the 'NIntegrate' command, to solve the improper integrals in equations (18) and (20). Firstly, we integrate equation (18) between 0 and $\varphi_0 - 10^{-12}$ (in order to avoid singularity) varying the values of φ_0 (the incognita). Using the Mathematica program, we obtained the values of φ_0 as a function of α . We chose as the error quota a result of equation (18) lower than 10^{-7} . This is very illustrative for the students, because they can see how it is possible to easily obtain the value of the incognita which is implicit in an improper integral. Figure 3 shows the results obtained for φ_0 as a function of α . We can see that, for low values of the load parameter ($\alpha < 0.25$), the deflections are small ($\varphi_0 < 15^\circ$). However, for large values, large deflections are obtained ($\varphi_0 \approx 50^\circ$ for $\alpha \approx 1.25$).

Knowing φ_0 as a function of α , it is easy to calculate the horizontal non-dimensional tip deflection ratio using equation (17), and integrating equation (18) with the aid of the Mathematica program, the vertical non-dimensional tip deflection ratio can also be calculated. Figures 4 and 5 show the results obtained. In figure 5 the value of β_y for small deflections has been included. It can be seen that the deflection β_y , obtained exactly from equation (18), and its approximate value calculated using equation (26) only coincide for small values of the load parameter α , that is, for small deflections of the cantilever beam. Finally, in figure 6 the elastic curves for different values of the load parameter have been plotted.

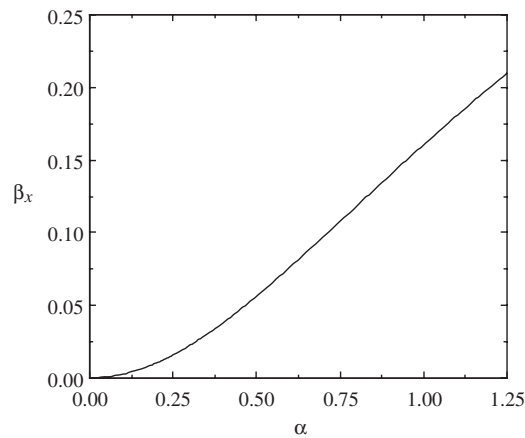


Figure 4. The horizontal non-dimensional tip deflection ratio, β_x , as a function of the non-dimensional load parameter α , for large deflections (for small deflections $\beta_x = 0$).

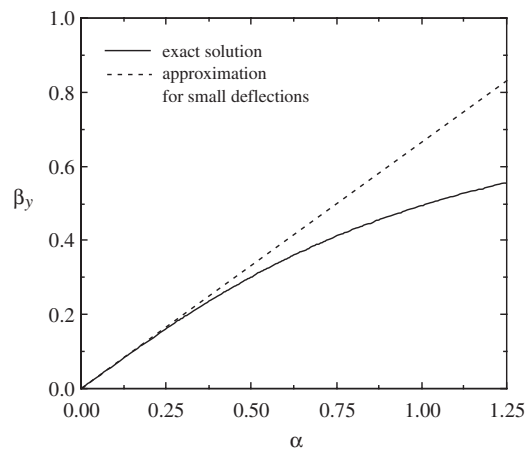


Figure 5. The vertical non-dimensional tip deflection ratio, β_y , as a function of the non-dimensional load parameter α , for large and small deflections.

4. Experimental results

In the laboratory it is possible to design simple experiments in order to analyse the deflection of a cantilever beam with a tip load applied at the free end [4]. For example, figure 7 shows a photograph of a system made up of a steel ruler of rectangular section built-in at one end and loaded at the free end with a mass. The length of the ruler is $L = 30$ cm and it has a uniform rectangular cross section of width $b = 3.04$ cm and height $h = 0.078$ cm. The moment of inertia of the cross section is given by the equation [2, 4, 9] $I = bh^3/12$ and its value is $I = 1.20 \times 10^{-12}$ m⁴. The weight of the ruler is $W = 0.554$ N and an external force $F = 3.92$ N acts on the free end of the ruler. The Young's modulus of the ruler material was calculated for small deflections and the value obtained was $E = 200$ GPa [4]. In the absence of an external load ($F = 0$), the maximum vertical deflection of the ruler, experimentally measured and theoretically calculated as $WL^3/8EI$ [2, 9], was approximately 7.8 mm. This means that the non-dimensional vertical deflection due to its own weight is approximately 0.026, which

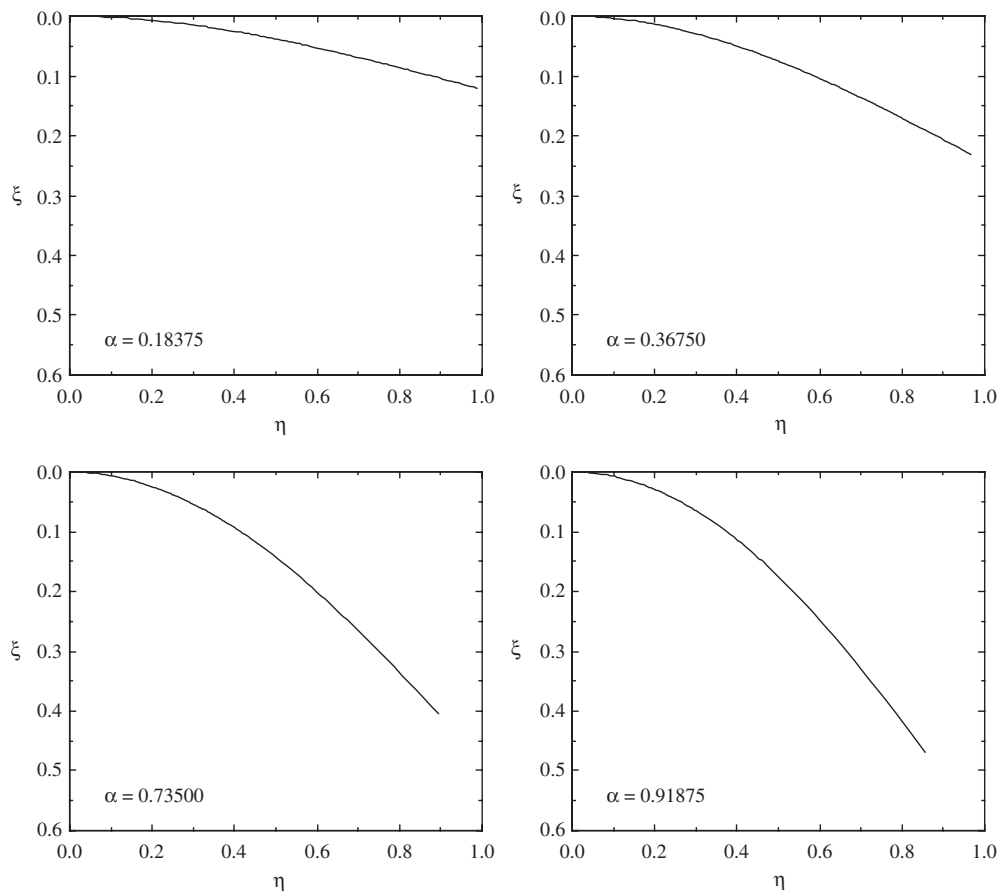


Figure 6. Elastic curves for different values of the load parameter.

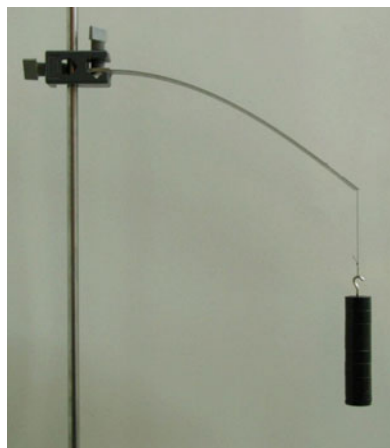


Figure 7. Photograph of a cantilever beam loaded with an external vertical concentrated load at the free end.

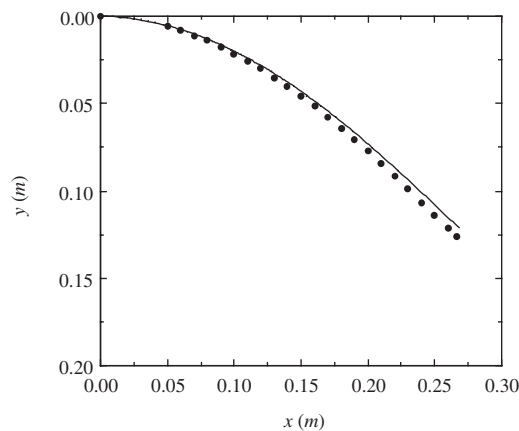


Figure 8. Experimental and theoretical elastic curves for the ruler analysed experimentally.

can be disregarded for large external loads. Taking into account the values of F , L , E and I and equation (15), the calculated value for the non-dimensional load parameter is $\alpha = 0.735$. From equations (16)–(20), we obtain $\varphi_0 = 0.62993$ rad (36.09°), $\beta_x = 0.1047$ ($\delta_x = 3.14$ cm) and $\beta_y = 0.4053$ ($\delta_y = 12.16$ cm). Figure 8 shows the experimental elastic curve as well as the one calculated with the aid of equations (18)–(20). The difference between both is due to the weight of the ruler itself, which has not been considered in the theoretical treatment.

5. Conclusions

We have studied the deflections of a cantilever beam both theoretically and experimentally. Firstly, we obtained the equations corresponding to the general case of large deflections and from these we deduced the equations corresponding to the case of small deflections. We have shown that, although we are dealing with a simple physical system, it is described by a differential equation with a non-linear term. Although the solutions to the elastic equations could be expressed in terms of elliptic functions, it is much more convenient for undergraduate students to use numerical integration as outlined in this paper. Numerical integration was performed using the Mathematica program to obtain the exact solution. Finally, we have shown that the deflections of a cantilever beam may be easily studied with a simple, easy-to-assemble, low-cost experiment, which allows us to experimentally study the deflections of cantilever beams by means of a series of simple measurements, such as of lengths and masses.

References

- [1] Alonso M and Finn E J 1992 *Physics* (Reading, MA: Addison-Wesley) ch 10
- [2] Feynman R, Leighton R B and Sands M 1989 *The Feynman Lectures on Physics: Mainly Electromagnetism and Matter* vol 2 (Reading, MA: Addison-Wesley) ch 38
- [3] Timoshenko S P 1983 *History of Strength of Materials* (New York: Dover)
- [4] Beléndez A, Neipp C and Beléndez T 2001 Experimental study of the bending of a cantilever beam *Rev. Esp. Fis.* **15** 42–5
- [5] Landau L D and Lifshitz E M 1986 *Course of Theoretical Physics: Theory of Elasticity* vol 7 (Oxford: Pergamon) ch 17
- [6] Lee K 2002 Large deflections of cantilever beams of non-linear elastic material under a combined loading *Int. J. Non-linear Mech.* **37** 439–43
- [7] Bisshopp K E and Drucker D C 1945 Large deflections of cantilever beams *Q. Appl. Math.* 272–5
- [8] McGill D J and King W W 1995 *Engineering Mechanics: Statics* (Boston: PWS) ch 5
- [9] Anderson H L (ed) 1989 *Physicist's Desk Reference: The Second Edition of Physics Vade Mecum* (New York: AIP) p 36